PMM U.S.S.R.,Vol.50,No.5,pp.576-583,1986
Printed in Great Britain

0021-8928/86 $10.00+0.00

SURFACE PROPERTIES AND STABILITY OF SHOCK WAVES IN GASES*

A.G. BASHKIROV

The boundary conditions on a curved shock wave are obtained from the laws of conservation of mass, energy, and momentum, ahead of the shock front, in the shock layer, and behind the shock front. They differ from the well-known conditions, allowing for the viscosity of the gas and heat conduction, in having extra terms, proportional to the shock-front curvature, the main extra term being the surface pressure with the surface tension. It must be said that, if the surface tension on the equilibrium inter-phase surfaces is linked with the anisotropy of the mean virial of the inter-molecular interaction force, then it is determined in the shock wave by the anisotropy of the viscous stresses in the shock layer and reaches a value 1 N/m at $M_0 = 10$ and 10 N/m at $M_0 = 30$. By taking account of surface tension when analysing the stability of a plane shock wave with respect to weak disturbance of the surface of discontinuity, we arrive at absolute instability of the mode of spontaneous sound radiation, which has previously been regarded as neutrally stable.

1. The boundary conditions on the discontinuity. The system of laws of conservation of mass, energy, and momentum densities throughout the gas volume, including the shock layer, is

$$\frac{\partial}{\partial t}\rho + \operatorname{div}\rho\mathbf{v} = 0, \quad \frac{\partial}{\partial t}\rho\mathbf{v} + \operatorname{Div}(p - \sigma + \rho\mathbf{v}\mathbf{v}) = 0$$
$$\frac{\partial}{\partial t}(u + {}^1/_2\rho v^2) + \operatorname{div}[(u + p - \sigma + {}^1/_2\rho v^2)\mathbf{v}] = -\operatorname{div} q \tag{1.1}$$

To describe the flow inside the shock layer we have to take account of the viscous stress $\sigma(\mathbf{x}, t)$ and the heat flux $\mathbf{q}$. In the incoming flow and the flow behind the shock front, the density $\rho(\mathbf{x}, t)$, velocity $\mathbf{v}(\mathbf{x}, t)$, pressure $p(\mathbf{x}, t)$, viscous stress tensor $\sigma(\mathbf{x}, t)$, internal energy density $u(\mathbf{x}, t)$ and head flux $q(\mathbf{x}, t)$ transform into the functions $\rho_0, v_0, p_0, u_0, \sigma_0 = q_0 = 0$ and $\rho_1, v_1, p_1, \sigma_1, u_1, q_1$ respectively, described by the equations of hydrodynamics of an ideal or viscous fluid. Inside the shock layer the profiles of these quantities can be obtained either in the Navier-Stokes approximation (for a weak shock wave), or by methods of the kinetic theory of gases.

Since the shock-wave structure will henceforth not be of independent interest, the differently scaled gas flow can be described approximately, by considering the shock layer as a surface of discontinuity.

To obtain the relations between the hydrodynamic parameters ρ_0, v_0, p_0, u_0 and $\rho_1, v_1, p_1, \sigma_1, u_1, q_1$ on the discontinuity, we locate a surface $\Sigma(t)$ inside the shock layer (the time dependence reflects the deformations of the discontinuity) and we define the excess values of the flow parameters $\rho^*, v^*, \ldots$ as

$$\rho^*(\mathbf{x}, t) = \int_{-\infty}^{\Sigma}[\rho(\mathbf{x}, t) - \rho_0]\,dz + \int_{\Sigma}^{\infty}[\rho(\mathbf{x}, t) - \rho_1]\,dz \tag{1.2}$$

where the z axis is along the normal $\mathbf{n}$ to Σ with respect to the gas flow.

By displacing $\Sigma(t)$ inside the shock layer, we can arrange for any one of the excess parameters to vanish. Below, for clarity, we position $\Sigma(t)$ in such a way that $v_n^* = 0$. If the profile $v_n(z)$ in the stationary shock wave is written approximately as

$$v_n(z) = {}^1/_2(v_0 + v_1) - {}^1/_2(v_0 - v_1)\operatorname{th}(2z/\delta) \tag{1.3}$$

where δ is the shock-layer thickness, then, in view of the equation $\rho(z)v_n(z) = \rho_0 v_0$, we obtain

*Prikl.Matem.Mekhan.,50,5,748-757,1986

from (1.2):

$$\rho^* = -{}^1/_4\rho_0\delta\,(v_0/v_1 - 1)\ln(v_0/v_1) \tag{1.4}$$

Applying the operation defined by (1.2) to the laws of conservation (1.1), we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\rho^* + \mathrm{div}_\tau(\rho v_\tau)^* = (v_{0n} - v_\Sigma)\rho_0 - (v_{1n} - v_\Sigma)\rho_1 - \\
&\quad (\rho v_n)^*\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \\
&\frac{\partial}{\partial t}(\rho \mathbf{v})^* + \mathrm{Div}_\tau(\rho\mathbf{v}\mathbf{v} - \sigma)^* + \nabla_\tau p^* = \mathbf{n}(p_0 - p_1 + \sigma_1) + \\
&\quad [\rho_0\mathbf{v}_0(v_{0n} - v_\Sigma) - \rho_1\mathbf{v}_1(v_{1n} - v_\Sigma)] - \frac{1}{R_1}(\rho v_n^2 - \rho v_{x1}^2)^* - \\
&\quad \frac{1}{R_2}(\rho v_n^2 - \rho v_{x2}^2)^* + \frac{1}{R_1}(\sigma_{nn}^* - \sigma_{11}^*) + \frac{1}{R_2}(\sigma_{nn}^* - \sigma_{22}^*) \\
&\frac{\partial}{\partial t}(u + {}^1/_2\rho v^2)^* + \mathrm{div}_\tau J_\tau^* = (u_0 + {}^1/_2\rho_0 v_0^2)(v_{0n} - v_\Sigma) - (u_1 + \\
&\quad {}^1/_2\rho_1 v_1^2)(v_{1n} - v_\Sigma) + p_0 v_{0n} - p_1 v_{1n} + \\
&\quad \sigma_{1nn}v_{1n} - q_{1n} - J_n^*\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \\
&J = (u + p + {}^1/_2\,\rho v^2 - \sigma)\,\mathbf{v} + \mathbf{q}
\end{aligned} \tag{1.5}$$

Here we have used the well-known expressions (see e.g., /1, p.29/) for the divergence of a vector and tensor in curvilinear coordinates z, x_1, x_2 connected with the surface Σ. The radii of curvature R_1 and R_2 are given in terms of the Lamé coefficients $H_z = 1,\ H_1,\ H_2$ by

$$\frac{1}{R_1} = \frac{1}{H_1}\frac{\partial H_1}{\partial z}, \quad \frac{1}{R_2} = \frac{1}{H_2}\frac{\partial H_2}{\partial z}.$$

The surface divergences are introduced by the relations

$$\mathrm{div}\,\mathbf{a} = \frac{\partial a_z}{\partial z} + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)a_z + \mathrm{div}_\tau\mathbf{a}_\tau$$

$$(\mathrm{Div}\ T)_z = \frac{\partial T_{zz}}{\partial z} + \frac{1}{R_1}(T_{zz} - T_{11}) + \frac{1}{R_2}(T_{zz} - T_{22}) + (\mathrm{Div}_\tau\ T)_z\,.$$

The surface laws of conservation (1.5) will be analysed for the special case of weak deformation of a plane normal shock wave:

$$z = \zeta(x, t) = \zeta\exp\{i(kx - \omega t)\}, \quad \frac{1}{R_1} = \frac{\partial^2\zeta}{\partial x^2} = -k^2\zeta\,.$$

While the general case of any surface $\Sigma(t)$ involves no difficulties in principle the working is laborious and unnecessary for the problem below of plane shock-wave stability.

We shall first estimate the terms on the left-hand sides of (1.5). Since all the excess densities are proportional to $\delta(M)$, where $M = M_0\cos(\partial\zeta/\partial x)$, then $\partial\rho^*/\partial t \sim \rho^* k^2\omega\zeta^2$, $\partial u^*/\partial t \sim u^* k^2\omega\zeta^2$, $\partial(\rho\mathbf{v}_\tau)^*/\partial t \sim \rho^* v_0 k\omega\zeta$.

It can be seen in a similar way that, of all the terms with space derivatives on the left-hand sides of (1.5), only $\mathrm{div}_\tau(\rho v_\tau)^* \approx -\rho^* v_0 k^2\zeta$ makes a contribution of the first order of smallness with respect to $k\zeta$.

We thus obtain from (1.5), in the linear approximation with respect to $k\zeta$, the following boundary conditions for the curved surface:

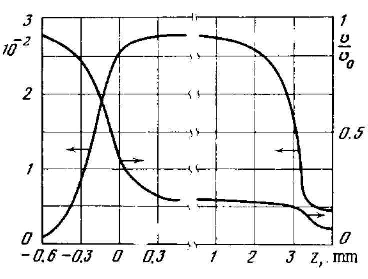

Fig.1

$$\begin{aligned}
&\rho_0(v_{0n} - v_\Sigma) = \rho_1(v_{1n} - v_\Sigma) - \rho^* v_0 k^2\zeta \\
&p_0 + \rho_0 v_{0n}(v_{0n} - v_\Sigma) = p_1 + \rho_1 v_{1n}(v_{1n} - v_\Sigma) - \sigma_{1nn} + p_\Sigma \\
&\rho_0 v_{0\tau}(v_{0n} - v_\Sigma) = \rho_1 v_{1\tau}(v_{1n} - v_\Sigma) - \sigma_{1n\tau} + \rho^* v_0 k\omega\zeta \\
&(u_0 + {}^1/_2\rho_0 v_0^2)(v_{0n} - v_\Sigma) + p_0 v_{0n} = (u_1 + {}^1/_2\rho_1 v_1^2)(v_{1n} - v_\Sigma) + \\
&\quad (p_1 - \sigma_{1nn})\,v_{1n} + q_{1n}
\end{aligned} \tag{1.6}$$

On the right-hand side of the second relation we have the surface pressure with surface tension

$$p_\Sigma = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\gamma, \quad \gamma = -(\sigma_{nn}^* - \sigma_{\tau\tau}^*) = -\int_{\infty}^{\infty}(\sigma_{nn} - \sigma_{\tau\tau})\,dz \tag{1.7}$$

($\sigma_{nn}(z)$, $\sigma_{\tau\tau}(z)$ are the viscous stresses in the shock layer).

2. Estimation of the surface tension. In the Navier-Stokes approximation we have from (1.7) to the zeroth order with respect to the shock-wave curvature:

$$\gamma_{NS} = -2\int_{-\infty}^{\infty} \mu(z)\frac{\partial v_n}{\partial z}\,dz \tag{2.1}$$

Putting $\mu(z) \sim (T(z))^{\alpha}$, and taking the dependence of $T(z)$ and $v_n(z)$ in the form (1.3), we find

$$\gamma_{NS} = \frac{2}{1+\alpha}\mu_0(v_0 - v_1)\frac{(T_1/T_0)^{\alpha}-1}{(T_1/T_0)-1} \approx \frac{2}{\alpha+1}\mu_1(v_0 - v_1) \tag{2.2}$$

In the presence of ionization of dissociation, the temperature and velocity characteristic profiles in the shock layer have the form given in /2/ and shown in Fig.1 ($M_0 = 30$, $p_0 = 1.0$ mm Hg, $T_0 = 300$ K, argon). Analytically these profiles can be written approximately as

$$U(z) = \frac{U_0 + U^{\circ}}{2} + \frac{U^{\circ} - U_0}{2}\,\mathrm{th}\,\frac{2z}{\delta_0}, \quad -\infty < z < z^{\circ}$$
$$U(z) = \frac{U^{\circ} + U_1}{2} + \frac{U_1 - U^{\circ}}{2}\,\mathrm{th}\,\frac{2(z-z_1)}{\delta_1}, \quad z^{\circ} < z < \infty,$$
$$z_1 \gg z^{\circ} > 0,$$

$U = vT$, $U^{\circ} = v^{\circ}$, T° are the values on the plateau of the relaxation zone. In this case we obtain from (2.1)

$$\gamma_{NS} \approx \frac{2}{\alpha+1}(v_0 - v_1)\,\mu(T^{\circ}) \tag{2.3}$$

It then follows from (1.7) that

$$\frac{\sigma_{1nn}}{\rho_1 v_1^2} \sim \frac{\mu_1}{\rho_1 v_1^2}\frac{\partial v_{1n}}{\partial z} \sim \frac{\mu_1}{\rho_1 v_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \varepsilon_R$$
$$\frac{p_{\Sigma}}{\rho_1 v_1^2} \sim \left(\frac{T^{\circ}}{T_1}\right)^{\alpha}\frac{v_0}{v_1}\varepsilon_R\,,$$

where ε_R is the inverse Reynolds number over the front curvature.

For the data of Fig.1, $v_0/v_1 \simeq 20$, $T^{\circ}/T_1 \simeq 6$. The exponent α depends on the intermolecular interaction and varies from 0.5 to 1.2, so that p_{Σ} can be two orders greater than σ_{1nn}.

A similar estimate can be obtained for a gas with oscillatory-rotational relaxation. According to kinetic theory /3/ of shock waves, nitrogen $v_0/v_1 \simeq 15$, $T^{\circ}/T_1 \simeq 3$ with $M_0 = 20$, so that p_{Σ} is one order greater then the viscous stress.

It is clear from these estimates that, when writing the boundary conditions /4/ on the discontinuity up to ε_R, account must be taken of p_{Σ}.

The above estimate of γ is obtained in the context of the hydrodynamic theory of shock-wave structure. We know /5/ that this theory satisfactorily describes the shock-layer structure only in the case of extremely low Mach numbers $M_0 \approx 1$, whereas, in the problem below, of shock-wave instability, we are interested in large M_0. However, γ is an integral characteristic of the layer, so that we can expect it to be satisfactorily described by the approximate hydrodynamic theory.

To prove this, we will calculate (2.3) for an ideal monatomic gas and compare the result with the calculation by the method of the kinetic theory.

In the case of a monatomic gas of rigid spheres

$$\mu_0 = \frac{5\pi}{32}\lambda_0\rho_0\sqrt{\frac{8kT}{\pi m}}, \quad \lambda_0 = \frac{m}{\sqrt{2}\pi r_0^2\rho_0} \tag{2.4}$$
$$\gamma_{NS} = \frac{25}{96}\sqrt{\frac{3\pi}{10}}\lambda_0 p_0\,\frac{(\eta - 16M_0^2)^{3/2} - 64M_0^3}{M_0^2(5M_0^2+3)},$$
$$\eta = 5M_0^4 + 30M_0^2 - 3$$

For small and large M_0, this expression transforms as follows:

$$\gamma_{NS}|_{M_0\to 1} = \frac{25}{8}\sqrt{\frac{3\pi}{10}}(M_0 - 1)\lambda_0 p_0 \approx 3{,}03\lambda_0 p_0(M_0 - 1) \tag{2.5}$$

$$\gamma_{NS}|_{M_0\to\infty} \approx 0{,}57\lambda_0 p_0 M_0^2 \tag{2.6}$$

We will consider the expression for the surface tension, obtained by the best known Tamm/Mott-Smith kinetic theory of plane normal shock-wave structure /6, 7/.
In this theory the molecular velocity distribution is the sum of two Maxwell distribution

functions

$$f(z) = f_0(z) + f_1(z)$$

$$f_\alpha(z) = n_\alpha(z)\left(\frac{m}{2\pi kT_\alpha}\right)^{3/2}\exp\left\{-\frac{m(\mathbf{c}-\mathbf{v}_\alpha)^2}{2kT_\alpha}\right\}, \quad \alpha = 0, 1 \tag{2.7}$$

where T_α and v_α are assumed constant, while the densities of the number of particles depend on the coordinates:

$$n_0(z) = \frac{n_0}{2}\left(1 - \operatorname{th}\frac{Bz}{2\lambda_0}\right), \quad n_1(z) = \frac{n_1}{2}\left(1 + \operatorname{th}\frac{Bz}{2\lambda_0}\right) \tag{2.8}$$

The coefficient B has the form* (*V.YU. Velikodnyi, Transport equations of multicomponent gas mixtures and strongly non-equilibrium gases, Candidate Dissertaion, MFTI, Moscow, 1982.)

$$B = \frac{24}{5}\sqrt{\frac{5}{3\pi}}\frac{(M_0^2-1)(\eta+(M_0^2-1)^2)}{(5M_0^2+3)(M_0^2+3)\eta^{1/2}}.$$

Using distribution function (2.7), we calculate the mean flow velocity, and the components of the pressure tensor $\Pi_{ii} = p - \sigma_{ii}$

$$v(z) = \frac{\langle c_z\rangle}{n_0(z)+n_1(z)} = \frac{n_0(z)v_0 + n_1(z)v_1}{n_0(z)+n_1(z)}$$

$$\Pi_{\tau\tau}(z) = m\langle c_x^2\rangle = m\langle c_y^2\rangle = n_0(z)kT_0 + n_1(z)kT_1 \tag{2.9}$$

$$\Pi_{nn}(z) = m\langle(c_z - v(z))^2\rangle = \Pi_{\tau\tau}(z) + m\frac{n_0(z)n_1(z)}{n_0(z)+n_1(z)}(v_0 - v_1)^2 \tag{2.10}$$

It is clear from this that, for a plane shock layer, kinetic theory, like Navier-Stokes theory, leads to the diagonal form of the pressure tensor in which the normal and tangential components are different.

Substituting (2.9) and (2.10) into definition (1.7) and integrating across the layer**, (**A.G. Bashkirov and G.A. Korol'kov, Surface tension of a shock-wave front, in: Aerophysics and geo-cosmic studies, MFTI, Moscow, 1983.) approximation is

$$\gamma_{MS} = \frac{25}{96}\sqrt{\frac{3\pi}{5}}\lambda_0 p_0\frac{(5M_0^2+3)(M_0^2+3)\eta^{1/2}}{\eta+(M_0^2-1)^2} \tag{2.11}$$

In the limiting cases of extremely weak and strong shock waves:

$$\gamma_{MS}|_{M_0\to 1} = \frac{25}{8}\sqrt{\frac{3\pi}{10}}\lambda_0 p_0(M_0 - 1) \approx 3{,}03\lambda_0 p_0(M_0 - 1) \tag{2.12}$$

$$\gamma_{MS}|_{M_0\to\infty} \approx 0{,}92\lambda_0 p_0 M_0^2 \tag{2.13}$$

Notice that (2.12) is exactly the same as (2.5), while (2.13) is less twice the size of (2.6). We can therefore assume that the Navier-Stokes theory corectly describes the shock layer integral characteristics (in order of magnitude), since the Mott-Smith theory agrees well with the experimental data in /5/ on shock-wave structure in monatomic gases.

With $T_0 = 300$ K we obtain from (2.6) and (2.13): $\gamma \sim 1$ N/m for $M_0 = 10$ and $\gamma \sim 10$ N/m for $M_0 = 30$. Numerical integration of the results in /8/ for $\Pi_{nn}(z)$ and $\Pi_{\tau\tau}(z)$, obtained by the Monte Carlo method with $M_0 = 8$ and $T_0 = 300$ K gives $\gamma \sim 1$ N/m. Similar processing of the experimental results of /9/ on the measurement of Π_{nn} and $\Pi_{\tau\tau}$ in a shock wave in helium with $M_0 = 1.59$ and $T_0 = 160$ K leads to $\gamma \sim 0.26$ N/m. These values agree well with our (2.4) and (2.11).

Such a large surface tension of the shock wave (recall for comparison that, under normal conditions, on the surface of water, $\gamma = 0.07$ N/m), is bound to affect its properties. In particular, they can explain the well-known experimental data on the excess pressure behind the front of a spherical shock wave /10/.

3. System of equations for small disturbances of discontinuity and flow.

In this and the next sections we shall first confine ourselves to the approximation of hydrodynamics and an ideal fluid for the gas behind the shock layer. We take a plane stationary shock wave of arbitrary intensity, satisfying the conditions $M_0 > 1$, $M_1 = v_1/c_1 < 1$. We specify a weak disturbance of the shock front as $\zeta(x, t) = \zeta\exp\{i(kx - \omega t)\}$, so that $v_\Sigma = -i\omega\zeta$. The shock-front disturbance generates harmonic disturbances of entropy δs, pressure δp, velocity $\delta\mathbf{v}$, and specific volume δV in the flow behind the shock front $(z > 0)$. They must satisfy the equations of ideal fluid hydrodynamics, which follow from (1.1), and the four linearized boundary conditions (1.6). The result is a homogeneous system of eight equations for the eight variables ζ, δp_2, δv_{1x}, δv_{1z}, δv_{2x}, δv_{2z}, δV_1, δV_2, where subscript 1 refers to the entropy disturbances, and 2 to the acoustic disturbances,

$$k\delta v_{1x}+(\omega/v_1)\,\delta v_{1z}=0,\quad \delta p_2+(c^2/V^2)\,\delta V_2=0 \tag{3.1}$$
$$(v_1 l-\omega)\,\delta v_{2x}+V_1 k\delta p_2=0,\quad (v_1 l-\omega)\,\delta v_{2z}+V_1 l\delta p_2=0$$
$$\delta v_{1x}+\delta v_{2x}-ik\,(v_0-v_1)\,\zeta+k\omega\rho^* V_0\zeta=0$$
$$\delta v_{1z}+\delta v_{2z}+\frac{v_0-v_1}{2}\left[\frac{\delta p_2+\delta p_\Sigma}{p_1-p_0}-\frac{\delta V_1+\delta V_2}{V_0-V_1}\right]=0$$
$$\frac{\delta p_2+\delta p_\Sigma}{p_1-p_0}+\frac{\delta V_1+\delta V_2}{V_0-V_1}-2i\frac{\omega}{v_0}\zeta=0$$
$$\delta p_2+\Phi\delta p_\Sigma-\left(\frac{\partial p}{\partial V}\right)_H(\delta V_1+\delta V_2)=0$$
$$\Phi=-\frac{3}{2}V_1\left[\frac{\partial w_1}{\partial p_1}-\frac{1}{2}(V_1+V_0)\right]^{-1}$$
$$\left(\frac{\partial p}{\partial V}\right)_H=-\left[\frac{\partial w_1}{\partial V_1}-\frac{1}{2}(p_1-p_0)\right]\left[\frac{\partial w_1}{\partial p_1}-\frac{1}{2}(V_1+V_0)\right]^{-1}$$

($(\partial p/\partial V)_H$ is the derivative along the ordinary Hugoniot shock adiabatic).

A detailed derivation of this system may be found in /11/. The main difference from the equations in /11/ is the appearance in the last three boundary conditions of the excess density and the surface pressure $\delta p_\Sigma=-\gamma k^2\zeta$.

4. Acoustic-entropy instability of the shock wave. For homogeneous linear system (3.1) to have a non-trivial solution, i.e., for non-zero amplitudes of the acoustic and entropy-viscous disturbances to exist, its characteristic determinant must vanish, whence we have the following characteristic equation (omitting here and henceforth the subscript 1)

$$\left(1+\frac{\omega^2}{h^2v^2}\right)\left(2\frac{\omega}{kv}-i\Phi\Gamma_\varphi\right)+\left(\beta-iP\frac{\omega}{kv}+\frac{\omega^2}{k^2v^2}\right)\times \tag{4.1}$$
$$\left(\frac{l}{k}-\frac{\omega}{kv}\right)(1+\varphi)-i\Gamma\left(\frac{\omega}{kv}\frac{l}{k}(1+\varphi)+1-\frac{\omega^2}{k^2v^2}\varphi\right)=0$$
$$P=-\frac{\beta\rho^* kv}{\rho_0(v_0-v)},\quad \Gamma=\frac{\beta\gamma k}{p-p_0},\quad \varphi=j^2\left(\frac{\partial V}{\partial p}\right)_H,\quad \beta=\frac{v_0}{v}$$

This equation differs from that obtained in /11/ in the terms with P and Γ, which take account of the surface properties of the shock wave.

Jointly with the dispersion equation, for the acoustic disturbances in the flow behind the shock wave,

$$(\omega-vl)^2=c^2(k^2+l^2) \tag{4.2}$$

Eq.(4.1) defines the dependence of ω and l on k and φ.

The condition for shock wave instability amounts to the existence of a solution which increases exponentially with time, and decreases exponentially as $z\to\infty$, i.e.,

$$\operatorname{Im}\omega>0,\quad \operatorname{Im} l>0 \tag{4.3}$$

Thus the problem of finding the domain of shock-wave instability amounts to finding the range of values of the parameter φ for which the solutions $\omega(k,\varphi)$ and $l(k,\varphi)$ of systems (4.1) and (4.2) satisfy both inequalities (4.3).

Let us first find the threshold values φ, corresponding to the boundary of the stability domain $\operatorname{Im}\omega=0$, $\operatorname{Im} l>0$; we shall then study the shock-wave stability on going beyond these boundaries.

In dimensionless variables $g+ih=-i\omega/(kv)$, $t+iu=il/k$, system (4.1), (4.2) becomes (with $g=0$)

$$t=-[\Gamma\chi+Ph\,(u+h)(1+\varphi)][(1+\varphi)(\beta+h^2)]^{-1} \tag{4.4}$$
$$2h\,(1+h^2)=(1+\varphi)(\beta+h^2)(h+u)+\Gamma th\,(1+\varphi) \tag{4.5}$$
$$M_1^2\,[t^2-(h+u)^2]=t^2-u^2-1,\quad M_1^2t\,(h+u)=tu \tag{4.6}$$
$$\chi=\Phi\varphi\,(1+h^2)+(1-h^2\varphi)-uh\,(1+\varphi)$$

By the second of conditions (4.3), $t=-\operatorname{Im} l/k<0$. This means that, for instability, we must have

$$\Gamma\chi+Ph(u+h)(1+\varphi)\gtrless 0,\quad \varphi\gtrless -1 \tag{4.7}$$

From the second of Eqs.(4.6) we find $u=M_1^2h/(1-M_1^2)$, after which it is clear from (4.5) that h can be zero.

With $h=0$, we obtain from the first of Eqs.(4.6), using (4.3): $t=-1/\sqrt{1-M_1^2}$. Jointly with (4.4), this gives the lower threshold of φ:

$$\varphi_H=-(1+\Gamma\sqrt{1-M_1^2})(1-\Gamma\Phi\sqrt{1-M_1^2})^{-1} \tag{4.8}$$

This expression differs from the lower bound of absolute stability $\varphi = -1$ obtained in /11/ by an amount of order Γ (i.e., of the order of the Knudsen number relative to the perturbation wavelength). The surface tension has virtually no effect on the shock-front stability with respect to a non-wave-type perturbation ($\mathrm{Re}\,\omega = -hkv = 0$). Hence, just as in the theory in /11/, for $\varphi < -1$ the shock wave becomes absolutely unstable.

For the threshold value φ_0 of instability with repect to a periodic perturbation ($h \neq 0$), we obtain from system (4.4)-(4.6):

$$\varphi_0 = \varphi_I + O(\Gamma^2), \quad \varphi_I = (1 - M_1^2 - \beta M_1^2)(1 - M_1^2 + \beta M_1^2)^{-1} \tag{4.9}$$

where φ_I is the same as the limit obtained earlier in /12, 13/ between the domain of shock-wave stability and the "neutrally stable" (with $\varphi_I < \varphi < 1 + 2M_1$) domain of spontaneous sound radiation.

Taking account of surface tension has virtually no effect on the position of the boundary of the domain of spontaneous sound radiation, though (see Sect.5 below) it makes this domain absolutely unstable.

From Eqs.(4.5) and (4.6), using the condition $\mathrm{Re}\,\omega > 0$, we find the remaining solutions

$$h_0 = -\sqrt{1 - M_1^2}/M_1, \quad u_0 = -M_1/\sqrt{1 - M_1^2} \tag{4.10}$$

Since $u = \mathrm{Re}\, l/k = \mathrm{ctg}\,\vartheta$ (ϑ is the angle between the acoustic wave, wave vector and the z axis), the expression obtained for u_0 reveals that spontaneous sound radiation begins (with $\varphi = \varphi_0$) from an angle $\vartheta_0 > \pi/2$ ($\cos\vartheta_0 = -M_1$). The vector l is here directed towards the flow, which "carried away" the sound wave /13/.

Substituting solution (4.10) into condition (4.7), we obtain the inequality

$$\Gamma\varphi(\Phi - 1) + P(1 + \varphi) > 0 \tag{4.11}$$

The second term is certainly greater than zero, while the first is greater than zero if

$$[\partial w/\partial p - (V_0 - 2V)/2]\,[\partial w/\partial V - (p - p_0)/2]^{-1} > 0 \tag{4.12}$$

Consequently, the necessary condition for the dstruction of stability (4.1) certainly holds if the factors in (4.12) are positive. From the definition of (3.1) for $(\partial p/\partial V)_H$ it is clear that, as v_0 increases, both these quantities (positive for small v_0) will remain positive throughout the forward course of the shock adiabatic, where it is continuous and $(\partial p/\partial V)_H < 0$. The denominator in (4.12) retains the positive sign even if the sign of $(\partial p/\partial V)_H$ changes, provided it passes through $\pm\infty$. At this point $\partial w/\partial p = (V_0 + V)/2$, so that the numerator of (4.12) remains positive in the neighbourhood of this point. Inequality (4.11) then holds.

Conversely, a change in the sign of $(\partial p/\partial V)_H$ on continuous passage through a local maximum of the shock adiabatic coresponds to a change in the sign of the denominator in (4.12), in which case (4.12) is violated and (4.11) may also be violated.

To prove the absolute instability of the spontaneous sound radiation mode, we put $\varphi = \varphi_0 + \Delta$ and consider system (4.1), (4.2) in the linear approximation with respect to Δ, noting that $\mathrm{Im}\,\omega \neq 0$ ($g \neq 0$). Then, instead of (4.4)-(4.6), we obtain the linear inhomogeneous system

$$a_i h' + b_i t' + c_i u' + d_i g = e_i \Delta, \quad i = 1, 2, 3, 4 \tag{4.13}$$

where the coefficients a_i, b_i, c_i, d_i, e_i are expressible in terms of the solution h_0, t_0, u_0, φ_0 of system (4.4)-(4.6), and the primes denote the deviations, linear in Δ, from these solutions. Hence

$$\mathrm{Im}\,\omega = kvg = -\frac{1 - M_1^2}{M_1^2(1 + \varphi_0)}\, t_0 kv\Delta \tag{4.14}$$

If condition (4.7) holds, then $t_0 < 0$ and $\mathrm{Im}\,\omega > 0$ for $\Delta > 0$, while $\mathrm{Im}\,\omega < 0$ for $\Delta < 0$. Hence the spontaneous sound radiation mode becomes absolutely unstable when the shock-wave surface tension is taken into account.

5. Influence of viscosity on shock-wave stability. When we take account of viscosity in the equations of hydrodynamics of the gas flow beind the shock front and in boundary conditions (1.5)-(1.7) and (1.11), extra terms appear in the linearized Eqs.(3.1), with the result that characteristic Eq.(4.1) takes the form

$$\left(1 + \frac{\omega^2}{k^2v^2}\right)\left(2\frac{\omega}{kv} - i\Phi\Gamma\varphi\right) + \left(\beta - iP\frac{\omega}{kv} + \frac{\omega^2}{k^2v^2}\right)\left(\frac{l}{k} - \frac{\omega}{kv}\right) \times$$
$$(1 + \varphi) - i\Gamma\left(\frac{\omega}{kv}\frac{l}{k}(1 + \varphi) + 1 - \frac{\omega^2}{k^2v^2}\varphi\right) - i\varepsilon\Psi\left(\frac{l}{k}, \frac{\omega}{kv}, \varphi, \varepsilon\right) = 0 \tag{5.1}$$
$$\varepsilon = \mu k/(\rho v), \quad \Gamma \sim \varepsilon (T^\circ/T)^\alpha \beta$$

where Ψ is a polynomial with real coefficients, and T° is the maximum temperature of the gas inside the shock layer.

The dispersion acoustic Eq. (4.2) in a viscous gas takes the form

$$\left(\frac{\omega}{kv}-\frac{l}{k}\right)^2+\frac{4}{3}i\varepsilon\left(\frac{\omega}{kv}-\frac{l}{k}\right)=\frac{1}{M_1^2}\left(1+\frac{l^2}{k^2}\right) \tag{5.2}$$

On the boundary of stability $(g=0)$, instead of (4.4)-(4.6) we have a system obtained by adding to the right-hand side of (4.4) a term proportional to $-\varepsilon\mathrm{Re}\Psi$, to the right-hand side of (4.5) a term $\varepsilon\,\mathrm{Im}\Psi$, and to (4.6) terms with ε from (5.2).

When analysing this new system, we note that, since the coefficients of the polynomial Ψ are real and $g=0$, we have $\mathrm{Im}\,\Psi\sim t\sim\varepsilon$, so that $\Gamma t\sim\varepsilon\,\mathrm{Im}\,\Psi\sim\varepsilon^2$ and they can be neglected. For the same reason, we can throw out the terms with εt.

We confine our study to the effect of viscosity on the boundary of the domain of spontaneous sound radiation, i.e., in the neighbourhood of solutions h_0, u_0 and φ_0, obtained above in the zero approximation with respect to ε. Instead of (4.10) and (4.9), we find

$$h_1\approx h_0,\quad u_1\approx-(1+\xi)\,u_0 \tag{5.3}$$

$$\varphi_1=\varphi_0\left(1-4\xi\,\frac{M_1^2}{1-M_1^2}\,\frac{1-(1+\beta/2)M_1^2}{1-(1+\beta)M_1^2}+2\xi\,\frac{M_1^2}{1-M_1^2}\right) \tag{5.4}$$

where $\xi={}^2/_3\,\varepsilon/t<0$. It can be shown that $|\xi|\ll 1$, in particular, $|\xi|<10^{-2}$ in argon and CO_2 with $\varphi\simeq\varphi_0$.

Since $M_1^2\ll 1$ and $|\xi|\ll 1$, these relations mean that, when account is taken of the effect of viscosity on the gas flow in the shock front there is a slight displacement of the boundary of the domain of spontaneous sound radiation $(\varphi_1>\varphi_0)$.

A study of the stability of the neighbourhood $\varphi=\varphi_1+\Delta$ amounts to analysing a system of equations similar to (4.13) except for terms $\sim\varepsilon$ in the coefficients b_1, d_1, a_2, c_2 and e_1, arising from the expansion of $\mathrm{Im}\Psi$ and $\mathrm{Re}\Psi$ in small disturbances h', t', u', Δ, g. As a result, for the imaginary part of the frequency, instead of (4.14), we have

$$\mathrm{Im}\,\omega=kvg=-\frac{1-M_1^2(1-\xi)}{M_1^2(1+\varphi_1)}\,t_1\Delta$$

where t_1 is given by a relation of type (4.4), to which is added a term with $-\varepsilon\mathrm{Re}\,\Psi$ in the case $h=h_1$, $u=u_1$, $\varphi=\varphi_1$.

Direct estimation of the explicit expression $\Psi(h_1, u_1, \varphi_1)$ gives $\mathrm{Re}\Psi>0$. We might expect from this and general considerations that the flow viscosity should strengthen the spatial damping of the acoustic perturbations.

Thus, $|t_1|>|t_0|$ and under condition (4.11) we certainly have $t_1<0$, whence follows the absolute instability of the shock-wave front with respect to generation of acoustic perturbations when $\Delta>0$, and absolute stability when $\Delta<0$.

This conclusion contradicts the result obtained in /14/, where it is shown that shock-wave stability is ensured by allowing for viscosity. However, the viscous damping of the shock-wave front perturbations obtained in /14/ is independent of the boundary conditions on the shock wave, and this is not physically justified. The reason seems to be that the viscous stresses on the curved surface were not quite correctly taken into account in /14/.

6. Discussion of the results. Our main result is the new boundary conditions (1.5) on the curved shock-wave, which include terms of first order in the inverse Reynolds number ε_R. By taking account of the shock-wave surface properties, extra terms appear, exceeding the viscous stresses introduced earlier in /4/ in the flow behind the shock front.

The use of new boundary conditions in the classical problem of plane shock-wave stability in a relaxing gas with respect to small perturbations of the surface of discontinuity leads to a new instability condition, quite different from those obtained in /11-13/. By taking account of surface effects, we arrive at a lowering of the boundary of the domain of spontaneous radiation, and what is most important, at absolute instability of this domain, which has previously been regarded as neutrally stable. This is explained physically by the excitation of capillary waves in the surface of the shock front and by their interaction with the radiated sound waves. This threshold of instability is accessible for shock waves in gases with dissociation or ionization.

It is difficult to compare the results with experimental data, inasmuch as the linear theory considered here only enables us to find the conditions under which the plane shock front becomes unstable, while nothing can be said about the final structure, whereas the coditions found experimentally are those under which a structure quite different from the plane shock front arises.

A series of experiments was made in /15/ to detect the instability of a plane shock wave in argon and CO_2. All the cases when stability is destroyed occurred with $\varphi\gtrsim\varphi_0$ on pieces of "backward run" of shock adiabatics (in CO_2 with $v_0>5$ km/sec and argon with $v_0>10$ km/sec), adjacent to the vertical pieces, and also on the piece of "forward run" (in CO_2 with $v_0\simeq 3.6$ km/sec). On all these pieces condition (4.11) holds, so that our present theory is in full agreement with the experiments of Griffiths et al.

REFERENCES

1. LOITSYANSKII L.G., Fluid and gas mechanics (Mekhanika zhidkosti i gaza), Nauka, Moscow, 1978.
2. CHUBB D.L., Ionizing shock structure in a monoatomic gas, Phys. Fluids, 11, 11, 1968.
3. PAI T.G. and RAMACHANDRA S.M., Structure of a shock wave in a vibrationally relaxing diatomic gas, Phys. Fluids., 18, 2, 1975.
4. SEDOV L.I., MIKHAILOVA M.N. and CHERNII G.G., On the influence of viscosity and heat conduction in gas flow behind a strongly curved shock wave, Vestn. MGU, Ser. Fiz.-mat. i est. Nauk, 2, 3, 1953.
5. ALSMEYER H., Density profiles in argon and nitrogen shock waves measured by the adsorbtion of an electron beam. J. Fluid Mech., 74, 3, 1976.
6. TAMM I.E., On the width of shock waves of high intensity, Tr. fiz. in-ta AN SSSR, 29, 1965.
7. MOTT-SMITH H.M., The solution of the Boltzmann equation for a shock wave, Phys. Rev., 82, 6, 1951.
8. BIRD G.A., Aspects of the structure of strong shock waves, Phys. Fluids, 13, 5, 1970.
9. MUNTZ E. and HARNETT L., Molecular velocity distribution function measurements in a normal shock wave, Phys. Fluids, 12, 10, 1969.
10. KARNEGAY W.M., FRIDMAN J.D. and WORTHINGTON W.C., Studies of spherical waves in low ambient pressures, In: Rarefield gas dynamics, Acad. Press, New York, 1969.
11. D'YAKOV S.P., On the stability of shock waves, Zh. Eksp. Teor. Fiz., 27, 3, 1954.
12. IORDANSKII S.V., On the stability of a plane stationary shock wave, PMM 21, 4, 1957.
13. KONTOROVICH V.M., On the stability of shock waves, Zh. Eksp. Teor. Fiz., 33, 6, 1957.
14. ZAIDEL' R.M., Development of disturbances in plane shock waves, Zh. Prikl. Mekh. Tekh. Fiz., 4, 1967.
15. GRIFFITHS R.W., SANDEMAN R.J. and HORNUNG H.G., The stability of shock waves in ionizing and dissociating gases, J. Phys., D, 9, 12, 1976.

Translated by D.E.B.

PMM U.S.S.R.,Vol.50,No.5,pp.583-594,1986
Printed in Great Britain

0021-8928/86 $10.00+0.00

ON INERTIAL EFFECTS ON DISCONTINUITIES IN THE CONCENTRATION OF THE SOLID PHASE IN A DISPERSE MEDIUM*

N.N. BOBKOV and YU.P. GUPALO

A system of conditions for the conservation of mass and momentum of a fluid and a solid phase on the surface of discontinuity in a disperse medium is analysed within the framework of the double continuum model /1/. The case when there is a high concentration of solid particles on one side of the discontinuity and a low or zero concentration of them on the other side is considered. Under these conditions in the region of a high concentration of the solid phase remote from the discontinuity, the inertial force of the fluid phase is small compared with the interphase interaction force and Darcy's law holds while both forces are of the same order of magnitude in the thin transition lower close to the surface of discontinuity. It is assumed that the surface of discontinuity is impermeable to the particles of the solid phase. Effects due to the possible occurrence of surface tension on the discontinuity are not considered.

Subject to the assumptions which have been made, a solution of the initial system of equations of motion and continuity of the phases is constructed in the stationary approximation taking account of the transition layer which satisfies the condition of the continuity of the

**Prikl.Matem.Mekhan.*,50,5,758-771,1986